ON STOCHASTIC CONVERGENCE OF THE SAMPLE EXTREME VALUES FROM DISTRIBUTIONS WITH INFINITE EXTREMITIES

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1. Introduction

Let $x_1, \ldots, x_n$ be a random sample of n units drawn from a population with a continuous cumulative distribution function (*cdf*) $F(x)$. For distributions, having infinite extremities, Fisher and Tippett (1928) considered the asymptotic distribution of $x_{(1)}$ and $x_{(n)}$, the sample smallest and the largest observations. Later, these findings were extended to the general case of $x_{(r)}$ and $x_{(n-r+1)}$, the r-th smallest and the r-th largest observations, for any given r, by Gumbel (1935). Two broad families of *cdf* were considered by them and termed the exponential and the Cauchy type of distributions, and the underlying regularity conditions were explicitly formulated by Von Mises (1936). The asymptotic distribution of the sample extreme value possesses a unique mode-termed the *characteristic extreme value* by Gumbel (1958).

Here we have considered the stochastic convergence of the sample extreme values to the corresponding characteristic extreme values, and the property is termed the consistency of the sample extreme values. This concept has been found to be very useful in establishing the consistency (or the inconsistency) of a class of multisample non-parametric tests by Mosteller (1948), Rosenbaum (1953), Kamat (1956), Haga (1960), among others and a detailed discussion of these has been made by the author, elsewhere (1961). Now the limiting distribution of $x_{(r)}$ or $x_{(n-r+1)}$ cannot by itself guarantee the consistency of it. In the particular case of $x_{(1)}$ (actually of $M_1 = \frac{1}{2}\{x_{(1)} + x_{(n)}\}$), Kendall and Stuart (1958, pp. 341) considered this property. They, however, have only considered two particular *cdf*'s, namely, the normal and the double exponential one, and have shown that, in the first case M_1 converges in probability to its population value, while in the second case, it does not. Also, their proof is based on the convergence (to zero) of the variance of M_1, which they have calculated using the limiting distribution of M_1. Now, it may be remarked that as for the consistency of a statistic, one need not bother about its variance, provided its sampling distribution is fairly specified—as is the case here. Secondly, the asymptotic value of the actual variance, may not necessarily, be equal to the variance of its limiting distribution, particularly, if the convergence is not in moments—a case which is also true for a subclass of exponential type of distributions, termed, the concave type (explained later on). So, here is established a general theorem on the consistency of the sample extreme values, avoiding these drawbacks and it has been shown that for the entire family of distributions of the Cauchy type, the sample extreme values as well as the extreme mid-ranges, are not consistent and even for the exponential type of distributions, they will be consistent, only under further extra-regularity conditions. Kendall and Stuart (1958, pp. 336) raised a further question as to the moments of the extreme values: whether the convergence of the moments of the sample extreme values to the corresponding ones of their asymptotic distribution follows along with the convergence of their distribution. Here, we have established this for a class of exponential type of distributions.

2. Notations and Preliminaries

For distributions of the exponential or the Cauchy type, the *cdf* $F(x)$ together with its first two derivatives $f(x)$ and $f'(x)$, is continuous everywhere, and

$$\lim_{x=\infty} [1 - F(x)] = 0, \quad \lim_{x=\infty} \{f(x) = F'(x)\} = 0$$

and

$$\lim_{x=\infty} f'(x) = 0; \quad \lim_{x=-\infty} F(x) = 0, \quad \lim_{x=-\infty} f(x) = 0,$$

and

$$\lim_{x=-\infty} f'(x) = 0. \tag{2.1}$$

Further, for large $|x|$, we have

$$\left.\begin{aligned} \frac{f(x)}{1-F(x)} &\doteqdot \frac{-f'(x)}{f(x)} \quad \text{for positive } x \\ \frac{f(x)}{F(x)} &\doteqdot \frac{f'(x)}{f(x)} \quad \text{for negative } x \end{aligned}\right\} \tag{2.2}$$

and thus they satisfy L' Hospital's rule. Let us also write

$$\left.\begin{aligned} \Phi_1(x) &= \frac{F(x)}{f(x)}, \quad C_1(x) = \frac{d\Phi_1(x)}{dx} \\ \Phi_2(x) &= \frac{[1 - F(x)]}{f(x)} \quad \text{and} \quad C_2(x) = -\frac{d\Phi_2(x)}{dx} \end{aligned}\right\} \tag{2.3}$$

Then the two families of *cdf* may be defined as follows.

(i) *Exponential type of* cdf.—Here $C_1(x)$ converges to zero as $x \to -\infty$ and $C_2(x)$ converges to zero as $x \to \infty$. It is further known that such a distribution possesses moments of all orders (*cf.* Gumbel, 1958, pp. 149).

(ii) *Cauchy type of* cdf.—In this case, we have

$$\left.\begin{aligned} &\lim_{x=\infty} C_2(x) = -\frac{1}{k_2} \quad \text{where} \quad k_2 > 0 \\ \text{and} \quad & \\ &\lim_{x=-\infty} C_1(x) = -\frac{1}{k_1} \quad \text{where} \quad k_1 > 0 \end{aligned}\right\} \tag{2.4}$$

Distributions of this class possess no moment of order: min. (k_1, k_2) or more (*cf.* Kendall and Stuart, 1958, pp. 333). Such distributions have also been considered by Fréchet (1927).

Let us also define the characteristic r-th largest value $x_{n.r}$ by $F(x_{n.r}) = (n - r)/n$ and the characteristic r-th smallest value ${}_{n,r}x$ by $F({}_{n.r}x) = r/n$ for $r = 1, 2, \ldots$ With these we will pass on to the study of the properties of the sample extremes.

3. Exponential Type of Distributions

We have from the definition of $\Phi_2(x)$, made in the preceding section,

$$\Phi_2(x) = \frac{[1 - F(x)]}{f(x)}, \quad i.e., \quad \frac{d\log(1 - F(x))}{dx} = -\frac{1}{\Phi_2(x)}.$$

Therefore, we get on integration

$$1 - F(x) = e^{-\psi_2(x)} \tag{3.1}$$

where

$$\frac{d\psi_2(x)}{dx} = \frac{1}{\Phi_2(x)}$$

and $\psi_2(\infty) = \infty$, $\psi_2(-\infty) = 0$. We now divide the class of exponential type of distributions into three broad categories:

(*a*) *Convex exponential type*.—In this case, in addition to $\lim_{x\to\infty} c_2(x) = 0$, we have $\lim_{x\to\infty} \Phi_2(x) = 0$ and a similar case follows with $\Phi_1(x)$ and $c_1(x)$, if the range extends to $-\infty$, on the left.

If, further, there exists a value of x, say x_1, such that for all $x \geqslant x_1$, $\Phi_2(x)$ is a monotonically decreasing function of x, then $F(x)$ will be termed a *strictly convex* exponential type. This additional restriction will only be required for the convergence of moments of the extreme values.

(*b*) *Simple exponential type*.—Here $\lim_{x\to\infty} \Phi_2(x) = d_2 > 0$ and a similar case with the lower extremity.

(*c*) *Concave exponential type*.—Here $\lim_{x\to\infty} \Phi_2(x) = \infty$, though it does so satisfying $\lim_{x\to\infty} c_2(x) = 0$. A similar case follows with the lower extremity.

Then we have the following:

THEOREM 3.1. For exponential type of distributions, the r-th largest (smallest) observation in a smaple of size n, converges in probability to the corresponding characteristic r-th largest (smallest) value, only for the convex type.

Proof.—In the neighbourbhood of $x_{n,r}$, the asymptotic distribution of $Z = x_{(n-r+1)}$ is given by (*cf.* Gumbel, 1935)

$$\left.\begin{aligned} g(u)\,du &= \frac{r^n}{\Gamma r} \exp.\,\{-ru - re^{-u}\}\,du \\ \text{where} & \\ u &= \frac{n}{r} f(x_{n,r})\,[Z - x_{n,r}] \end{aligned}\right\} \quad (3.2)$$

Also, by definition of $\Phi_2(x)$, we have

$$\frac{n}{r} f(x_{n,r}) = [\Phi_2(x_{n,r})]^{-1} \quad (3.3)$$

Now, as $F(x_{n,r}) = (n-r)/n$ and as the range is extended to ∞, we have from the monotonicity of $F(x)$ that $\lim_{n=\infty} x_{n,r} = \infty$. Thus, it follows from (3.3) that

$$\left.\begin{aligned} \lim_{n=\infty} \frac{n}{r} f(x_{n,r}) &= \infty, \quad \text{if } F(x) \text{ is a convex exponential type of } cdf \\ &\leqslant \frac{1}{d_2}, \quad \text{for non-convex exponential type of } cdf, \text{ where } d_2 < \infty. \end{aligned}\right\} \quad (3.4)$$

Let us first consider the convex exponential type of *cdf* and consider any sequence of functions $\{C_n = H(x_{n.})\}$, such that

$$\lim_{n=\infty} \epsilon_n = 0, \text{ but } \lim_{n=\infty} \left[\frac{\epsilon_n}{\Phi_2}(x_{n,r}) = b_n)\right] = \infty, \quad (3.5)$$

this being always possible as $\lim_{n=\infty} \Phi_2(x_{n,r}) = 0$. We have then for large n,

$$\text{Prob.}\{x_{n,r} - \epsilon_n \leqslant Z \leqslant x_{n,r} + \epsilon_n\} = I_r(re^{b_n}) - I_r(re^{-b_n}), \quad (3.6)$$

where

$$I_r(x) = \frac{1}{|\overline{r}} \int_0^x e^{-x} x^{r-1}\, dx.$$

Now, as $\lim_{n=\infty} b_n = \infty$, we get from the well-known properties of the incomplete Gamma functions that

$$\lim_{n=\infty} I_r(re^{-b_n}) = \lim_{n=\infty} \{1 - I_r(re^{b_n})\} = 0. \quad (3.7)$$

Hence, from (3.6) and (3.7), we get

$$\lim_{n=\infty} \text{Prob.}\{|x_{(n-r+1)} - x_{n,r}| \leqslant \epsilon_n\} = 1.$$

Thus, $x_{(n-r+1)}$ converges in probability to $x_{n,r}$.

For non-convex exponential *cdf*'s, we note that whatever be the sequence $\{\epsilon_n\}$ of arbitrarily small positive quantities ϵ_n, with $\lim_{n=\infty} \epsilon_n = 0$, we would have [as follows from (3.4)]:

$$\lim_{n=\infty} \left\{b_n = \frac{\epsilon_n}{\Phi_2(x_{n,r})}\right\} = 0, \quad (3.8)$$

and hence from (3.6), we get after simple manipulations, [using (3.6)],

$$\lim_{n=\infty} \text{Prob.}\{|x_{(n-r+1)} - x_{n,r}| \leqslant \epsilon_n\} = 0.$$

Thus $x_{(n-r+1)}$ cannot converge in probability to $x_{n,r}$. A similar result applies to the lower extreme values.

Hence, the theorem.

Cramér (1946, pp. 376), following the lines of Fisher and Tippett (1928) and Gumbel (1935), has suggested that the asymptotic distribution in (3.2) also holds for the class of *cdf*'s, for which, we have, for large $| x |$, at least

$$f(x) = A \exp. \{- B \mid x \mid^{p}\} \quad \text{where} \quad A, B, p > 0.$$

It now follows from our Theorem 3.1 that only for $p > 1$, the sample extreme values will be consistent in the above sense, while for $p \leqslant 1$, they will not be so. Also, of the two examples considered by Kendall and Stuart (1958, pp. 341), the normal *cdf* belongs to the family of convex exponential type, while the double exponential belongs to the family of simple exponential type; hence, in the first case, the extreme values (and consequently the mid-ranges) converges, while in the later one, they do not.

As the entire class of exponential type of *cdf*'s possesses moments of all finite orders (*cf.* Gumbel, 1958, pp. 149), it follows from a theorem on the existence of the moments of order statistics (Sen, 1959, Theorem 2.1) that the same is also true for the distributions of the extreme values. In most of the cases, these moments cannot be evaluated by direct integration and as the labour involved in the quadrature procedure increases tremendously with the sample size, one is naturally inclined to use some simple and valid approximations, at least for the large samples. And, here is considered the problem of the convergence of the moments of the extreme values to the corresponding ones of their limiting distribution, as sketched in (3.2). Here is established this for both convex and simple exponential types, while for concave type, we have not been able to do this, nor we think it to be easily approachable.

THEOREM 3.2. For strictly convex exponential type of distributions, the k-th moment of the r-th extreme value (in a sample of size n) about its characteristic extreme value, converges asymptotically to the corresponding one, derived from its limiting distribution, given in (3.2).

Proof.—We are to show that

$$\lim_{n \to \infty} E \left\{ \left[\frac{n}{r} f(x_{n,r}) \right] \left[x_{(n-r+1)} - x_{n,r} \right] \right\}^{k} = \mu_k^0 \qquad (3.9)$$

where $\mu_k{}^0$ is the k-th moment (about the origin) of u [defined in (3.2)] and the expression for $\mu_k{}^0$ is available with Kendall and Stuart (1958, pp. 336).

Let us now denote by $g(Z)$, the density function of $Z = x_{(n-r+1)}$. Then

$$g(Z)\,dZ = r\binom{n}{r}[F(Z)]^{n-r}[1-F(Z)]^{r-1}\;dF(Z).$$

Let us also define $x_{1(n)}$ and $x_{2(n)}$ by

$$x_{1(n)} = x_{n,r} - \epsilon_n \quad\text{and}\quad x_{2(n)} = x_{n,r} + \epsilon_n, \tag{3.10}$$

where ϵ_n has already been defined in the proof of Theorem 3.1. Then we can write

$$E(Z - x_{n,r})^k = \left[\int_{-\infty}^{x_{1(n)}} + \int_{x_{1(n)}}^{x_{2(n)}} + \int_{x_{2(n)}}^{\infty}\right](Z - x_{n,r})^k\, g(Z)\,dZ$$
$$= I_1 + I_2 + I_3 \text{ (say)}. \tag{3.11}$$

Since $x_{n,r}$ is an increasing function of n with $\lim\limits_{n\to\infty} x_{n,r} = \infty$, there exists a value of n, say n_0, such that for all $n \geqslant n_0$, $x_{n,r} \geqslant x_1$, where for $x \geqslant x_1$, $\psi_2(x)$ is a convex function of x. Taking then n adequately large, we have after some simple computations

$$|I_3| = r\binom{n}{r}\int_{x_{2(n)}}^{\infty}(Z - x_{n,r})^k\, g(Z)\,dZ$$
$$\leqslant \frac{r^r}{\lfloor r}[\Phi_2(x_{n,r})]^k \int_{\psi_2(x_{2(n)})-\log n/r}^{\infty} y^k e^{-ry}\left(1 - \frac{r}{n}e^{-y}\right)^{n-r} dy. \tag{3.12}$$

Thus, it can be shown with simple algebraic manipulations that

$$\lim_{n\to\infty}\left[\frac{n}{r}f(x_{n,r})\right]^k |I_3|$$
$$= \lim_{n\to\infty}[\Phi_2(x_{n,r})]^{-k}|I_3|$$
$$\leqslant \lim_{n\to\infty}\frac{r^r}{\lfloor r}\int_{b_n}^{\infty} e^{-ry-re^{-y}}\, y^k dy = 0. \tag{3.13}$$

[where b_n has been defined in (3.5)].

Again

$$I_1 = \int_{-\infty}^{x_{1(n)}} (Z - x_{n,r})^k \, g(Z) \, dZ$$

$$= \left[\int_{-\infty}^{x_1} + \int_{x_1}^{x_{1(n)}} \right] (Z - x_{n,r})^k g(Z) \, dZ$$

$$= I_{11} + I_{12} \text{ (say).}$$

Then

$$| I_{11} | \leqslant r \binom{n}{r} [c_{2k} \{ \nu_{2k} + | x_{n,r} |^{2k} \}]^{1/2} . p_1^{n-r+1/2} (1 - p_1)^{r-1} \tag{3.14}$$

where $0 < p_1 = F(x_1) < 1$, $c_k = 2^{k-1}$ for $k > 1$ and $c_k = 1$ for $k \leqslant 1$; ν_k is the absolute k-th moment of the density function $f(x)$. Also, it follows from the convexity of $\psi_2(x)$ for $x \geqslant x_1$, that

$$x_{n,r} - x_1 \leqslant \Phi_2(x_1) [\psi_2(x_{n,r}) - \psi_2(x_1)],$$

i.e.,

$$x_{n,r} \leqslant a + b \log \frac{n}{r} \tag{3.15}$$

where a and b are finite for any n. Thus, in view of the following $\lim_{n=\infty} n^{r+k} (\log n)^k p^n = 0$ for any $0 < p < 1$ and given k, r, we get from (3.14) and (3.15) that

$$\lim_{n=\infty} \left| \left[\frac{n}{r} f(x_{n,r}) \right]^k I_{11} \right| = 0. \tag{3.16}$$

It can similarly be shown with lengthy algebraic computations (the details of which are available with the author, 1961) that

$$\left[\frac{n}{r} f(x_{n,r}) \right]^k . | I_{12} |$$

$$\leqslant \frac{r^r}{\lceil r} [\Phi_2(x_{n,r})]^{-k} c^k \int_{b_n}^{\infty} y^k e^{ry} \left(1 - \frac{r}{n} e^y \right)^{n-r} dy \tag{3.17}$$

(where c is a finite positive quantity) and hence after some manipulations we get that

$$\lim_{n=\infty} \left[\frac{n}{r} f(x_{n,r}) \right]^k | I_{12} | = 0 \text{ for any given } k. \tag{3.18}$$

From (3.13), (3.16) and (3.18) we get, at once,

$$\lim_{n\to\infty}\left[\left\{\frac{n}{r}f(x_{n.r})\right]^k E(Z-x_{n.r})^k\right\}$$
$$=\lim_{n\to\infty}\left\{\left[\frac{n}{r}f(x_{n.r})\right]^k I_2\right\} \tag{3.19}$$

w,

$$\left[\frac{n}{r}f(x_{n.r})\right]^k I_2$$
$$=\frac{r\binom{n}{r}}{[\Phi_2(x_{n.r})]^k}\int_{\psi_2(x_{1(n)})}^{\psi_2(x_{2(n)})}(Z-x_{n.r})^k e^{-r\psi_2(Z)}(1-e^{\psi_2(Z)})^{n-r}d\psi_2(Z). \tag{3.20}$$

Also for all $x_{1(n)}=x_{n.r}-\epsilon_n\leqslant x\leqslant x_{n.r}+\epsilon_n=x_{2(n)}$, we have

$$\psi_2(Z)-\psi_2(x_{n.r})=\frac{(Z-x_{n.r})}{\Phi_2\{x_{n.r}+\lambda(Z-x_{n.r})\}},\quad 0<\lambda<1$$

and further, it can be shown with little difficulty, that for all such x,

$$\frac{\Phi_2(x_{n.r}+\lambda\epsilon_n)}{\Phi_2(x_{n.r})}=1+\delta_n \tag{3.21}$$

where $\delta_n\to 0$ as $n\to\infty$. Hence, (3.20) reduces, after some simplification, to

$$\frac{r^r}{\Gamma r}\int_{-b_n}^{b_n} y^k e^{-ry-re^{-y}}\,dy+\eta_n \tag{3.22}$$

where $\eta_n\to 0$ as $n\to\infty$. Now as $b_n\to\infty$ with $n\to\infty$, we get from the convergence of the integral in (3.22) at $\pm\infty$, that

$$\lim_{n=\infty}\left[\frac{n}{r}f(x_{n.r})\right]^k I_2=\mu_k^0. \tag{3.23}$$

Hence, from (3.19) and (3.23), we get that

$$\lim_{n=\infty}\left[\binom{n}{r}f(x_{n.r})\{x_{(n-r+1)}-x_{n.r}\}\right]^k=\mu_k^0.$$

A similar case follows with $x(r)$. Hence, the theorem.

THEOREM 3.3. For simple exponential type of distributions, the moments of the sample extreme values, about their characteristic extreme values, converges to the corresponding ones of their limiting distribution.

The proof is simple. It can be shown that for such distributions, we have for all large n and for all finite $Z = n[1 - F(x)]$,

$$\frac{n}{r}[1 - F(x)] = \exp. \left\{ - \frac{(x - x_{n.r})}{d_2[1 + \eta_n^{(1)}]} \right\} \tag{3.24}$$

where $\eta_n^{(1)} \xrightarrow{p} 0$. With this we can proceed similarly as in the preceding theorem and arrive at the desired result.

It thus follows from those two theorems that for such exponential type of distributions, we can use the limiting distribution of the sample extreme values, for large samples at least, for evaluating their central moments.

4. Cauchy Type of Distributions

Let us first consider the distribution of $x_{(r)}$ or $x_{(n-r+1)}$, for large samples and for that, we apply the same technique, as applied by Gumbel (1935) for the case of exponential distributions. The distribution of Z is given by (where $Z = x_{(n-r+1)}$)

$$g(Z)\,dZ = r\binom{n}{r}[(F(Z)]^{n-r}[1 - F(Z)]^{r-1} f(Z)\,dZ. \tag{4.1}$$

We now expand $F(Z)$ locally about $F(x_{n.r}) = (n - r)/n$ and get

$$F(Z) = F(x_{n.r}) + (Z - x_{n.r}) f_{(x_{n.r})} + \frac{(Z - x_{n.r})^2}{2!} f'(x_{n.r}) + \ldots \tag{4.2}$$

Now from the condition $\lim\limits_{x\to\infty} c_2(x) = -1/k_2$, we get

(i) $\lim\limits_{x\to\infty} \dfrac{x f(x)}{[1 - F(x)]} = k_2,$

(ii) $f(x_{n.r}) = \dfrac{k_2[1 - F(x_{n.r})]}{x_{n.r}}$, and

(iii) $f'(x_{n.r}) = -\dfrac{k_2(k_2 + 1)[1 - F(x_{n.r})]}{x^2_{n.r}}$, etc.

With these, we get from (4.1) and (4.2), after some simple adjustments, that in the neighbourhood of $x_{n.r}$, the large sample distribution of Z comes out as

$$g(Z)\,dZ = \frac{r^r}{\lceil r}\left(\frac{Z}{x_{n,r}}\right)^{-(k_2 r-1)} \exp.\left\{-r\left(\frac{Z}{x_{n,r}}\right)^{-k_2}\right\} \cdot \frac{k dZ}{x_{n,r}}, \quad (4.3)$$

and in the particular case of $r = 1$, this tallys with the expression given by Fisher and Tippet (1928). We have then the following:

THEOREM 4.1.—For the entire class of Cauchy type of distributions, the sample extreme values do not converge in probability to their characteristic extreme ones.

Proof.—Let us consider the case of $Z = x_{(n-r+1)}$. Then, for any given sequence $\{\epsilon_n\}$ of arbitrarily small positive quantities, with $\lim_{n\to\infty} \epsilon_n = 0$, we have from (4.1) and (4.3)

$$\text{Prob. } \{| Z - x_{n,r} | \leqslant \epsilon_n\} \doteqdot \frac{r^r}{\lceil r} \int_{a_n}^{b_n} e^{-y}\, y^{r-1}\, dy$$

where

$$a_n = r\left\{1 + \frac{\epsilon_n}{x_{n,r}}\right\}^{-k_2} \text{and } b_n = r\left\{1 - \frac{\epsilon_n}{x_{n,r}}\right\}^{-k_2}. \quad (4.4)$$

Now as,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = r, \text{ we get from (4.4) that}$$

$$\lim_{n\to\infty} \text{Prob. } \{| Z - x_{n,r} | \leqslant \epsilon_n\} = 0. \quad (4.5)$$

The case with $x_{(r)}$ follows similarly.

Hence, the theorem.

In view of the inconsistency of the sample extreme values, little has to be done with them, as regards their applications in the theory of inference. Further, it can be shown that for such distributions, moments of the sample extreme values, higher than a certain order, do not exist (*cf.* Sen, 1961). So, for such distributions, the problem of the convergence of the moments (only which actually exist) has not been considered.

We have so far considered the case of sample extreme values. The case with the extreme mid-ranges and ranges, their consistency (in the same sense) and asymptotic convergence of moments will follow precisely on the same line and hence need not be reproduced here.

5. Summary

In this paper, the stochastic convergence of the sample extreme values to the corresponding characteristic extreme values, as well as the asymptotic convergence of their central moments to the corresponding ones of their limiting distributions, have been studied, for parent distributions of the exponential or the Cauchy type. These findings appear to be very useful in studying the consistency and the asymptotic power-efficiency of a class of multi-sample non-parametric tests, based on the number of rare exceedances.

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7. References

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